THE SHAPE OF THE SIDES OF AN ARC

BY
JAMES M. SLOSS

ABSTRACT

Associated with certain oriented Jordan arcs is a region which is called the side of the arc. Circles centered on the arc and passing through one of the end points of the arc have an envelope which permits one to find analytically the shape of the side of the arc. A condition, stated geometrically, is imposed on the arcs considered, to insure that the circles have an envelope. An analytic condition is then imposed (Theorem 3) to insure an envelope. As an example a parabola is given.

In this note we shall be concerned with finding the shape of the left and right neighborhoods of a Jordan arc L with a prescribed direction, as described in Muskhelishvili [1].

We define the left neighborhood (or left side) S^+ of an arc L with end points L_a and L_b as follows: a point p is in S^+ if there exists a point t_0 on L and an open disc D_0 centered at t_0 for which $D_0 - L$ is the union of two disjoint open sets, one on the left, D_0^+, and one on the right, D_0^-, as we traverse L in the prescribed direction, and p is in D_0^+. The right neighborhood S^- is defined in an obvious way. It will be our aim to find the shape of S^+.

In this note we shall impose a further restriction on L, namely: (C_1) every circle centered on L and passing through L_a or L_b is divided into at most two disjoint sets by L. Note that the arc $y = 2\sin \pi x$, $0 \leqq x \leqq 1$ does not satisfy (C_1) since the circle centered on L which is tangent to the line $x = 1$ and passes through $(0,0)$ is cut by L in four points.

We now define the set E^+. A point p is in E^+ if there exists a point t_1 and an open disc D_1 centered at t_1, whose closure passes through L_a or L_b, $D_1 - L$ is the union of two disjoint open sets D_1^+ and D_1^-, for which p is in D_1^+.

THEOREM 1. $S^+ = E^+$.

Proof. $S^+ \supseteq E^+$ clearly. To see that $E^+ \supseteq S^+$ let p be in S^+ and let the disc containing p, centered at t_0 on L, and divided by L into two disjoint sets, be denoted by D_0. Let the circular boundary of D_0 be denoted by B_0.

We shall show as a preliminary result that D_0 cannot contain either L_a or L_b. To see this, assume that D_0 contained L_a or L_b. Then since D_0 is divided into two disjoint sets by the non-intersecting continuous arc L, L must intersect B_0 in at least two points b_1 and b_2. Without loss of generality, assume b_1 lies between

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t_0 and L_a on L. Since L_a is an interior point of D_0, there is a circle about t_0, smaller than B_0, which contains L_a in its interior, and b_1 and b_2 in its exterior. Such a circle is cut by L in at least three distinct points and thus (C_1) is violated and we see that D_0 cannot contain L_a or L_b.

If B_0 contains L_a or L_b we are finished, since then p is in E^+. On the other hand, if B_0 contains neither L_a nor L_b, we know from the above paragraph that $D_0 \cup B_0$ contains neither L_a nor L_b. Increase the size of D_0 (maintaining its center at t_0) until B_0 cuts L_a or L_b or both. Then D_0 becomes a D_1 with $D_0^+ \subseteq D_1^+$ and thus p is in E^+. Thus $E^+ = S^+$.

We shall now introduce a second condition: (C_2) Let L be a Jordan arc with end points L_a and L_b. Given any point t on L there exists a circle $\gamma_t(\varepsilon)$ centered at t and of radius $\varepsilon > 0$ such that the extended line joining any point of L interior to $\gamma_t(\varepsilon)$ and passing through t does not contain L_a or L_b. Note that the interval $[0,1]$ does not satisfy (C_2).

Definition. The class of arcs that satisfy (C_1) and (C_2) and which have a continuously turning tangent will be denoted by $\mathscr{L}$.

Theorem 2. *If $L \in \mathscr{L}$, the circles centered on L and passing through $L_a(L_b)$ have an envelope.*

Proof. This follows since any two such circles, if sufficiently close together, intersect in one other point than L_a since if not they would be tangent and their centers would lie on a line through L_a. But this is impossible since $L \in \mathscr{L}$. A similar argument holds for circles passing through L_b.

The condition (C_2) is defined geometrically. We shall now be concerned with finding a subset of $\mathscr{L}$ that is characterized analytically.

Lemma. *Let $g(s)$ be a continuous map from $[\sigma^*, \sigma^{**}]$ to L^* and let $g(s_0) \neq 0$ for $\sigma^* < s_0 < \sigma^{**}$. Then there is a unit vector A and a circle $\gamma_{s_0}(\varepsilon^*)$ about $g(s_0)$ of radius $\varepsilon^* > 0$ such that $g(s)$ is not perpendicular to A for $g(s)$ in $\gamma_{s_0}(\varepsilon^*)$.*

Proof. By contradiction. Let $\varepsilon_n \to 0$ as $n \to \infty$. Then if A is an arbitrary, but fixed unit vector, there is an s_n with $g(s_n)$ in $\gamma_{s_0}(\varepsilon_n)$ such that $g(s_n)$ is perpendicular to A and thus $g(s_n) \cdot A = 0$. Since the $\varepsilon_n \to 0$ it follows that $g(s_n) \to g(s_0)$ and thus $g(s_n) \cdot A \to g(s_0) \cdot A$. But for each n, $g(s_n) \cdot A = 0$ and thus $g(s_0) \cdot A = 0$. But A was arbitrary and $|A| = 1$. Thus $g(s_0) = 0$ which is a contradiction.

Theorem 3. *Let the Jordan arc L: $y = y(s)$, $0 \leqq s \leqq \sigma$, $s =$ arc length of L, satisfy (C_1). If for every $s_0, 0 < s_0 < \sigma$, there exists a positive integer $n(s_0) \geqq 2$ such that $y(s)$ has n continuous derivatives in a neighborhood of $s = s_0$, $y'(s_0) \neq 0$, $y''(s_0) = \dots = g^{(n-1)}(s_0) = 0$ but $y^{(n)}(s_0) \neq 0$, then L belongs to $\mathscr{L}$.*

Remark. This theorem says geometrically that if L has some "curviness" at each point then L satisfies (C_2).

Proof. We shall first show there is a circle $\gamma_{s_0}(\varepsilon)$ centered at $y_0 \equiv y(s_0)$ of such small radius $\varepsilon > 0$ that no radius of $\gamma_{s_0}(\varepsilon)$ intersects L within $\gamma_{s_0}(\varepsilon)$ in more than n points. Since $y^{(n)}(s)$ is continuous at $s = s_0$ there is a δ, $0 < \delta < < \min\{s_0, \sigma - s_0\}$ such that if $|s - s_0| < \delta$, then

$$|y^{(n)}(s) - y_0^{(n)}| < |y_0^{(n)}|, \text{ where } y_0^{(n)} = y^{(n)}(s_0)$$

since $y_0^{(n)} \neq 0$. Let

$$\varepsilon^{**} = \min\{|y(s_0 + \delta) - y_0|, |y_0 - y(s_0 - \delta)|\}.$$

Apply the lemma to $g(s) = y^{(n)}(s)$ and get an ε^* and an A, $|A| = 1$ such that $y^{(n)}(s)$ is not perpendicular to A for $y^{(n)}(s)$ in $\gamma_{s_0}(\varepsilon^*)$. Let $\varepsilon = \min\{\varepsilon^*, \varepsilon^{**}\}$.

Let $\delta^* > 0$ be such that $y(s_0 - \delta^*)$ is the first point of intersection of L with $\gamma_{s_0}(\varepsilon)$ as we travel along L from y_0 to L_a and $\delta^{**} > 0$ be such that $y(s_0 + \delta^{**})$ is the first point of intersection of L with $\gamma_{s_0}(\varepsilon)$ as we travel along L from y_0 to L_b. Without loss of generality, we may assume that $y^{(n)}(s)$ is defined and continuous on $I_0 = (s_0 - \delta^*, s_0 + \delta^{**}) \subset [0, \sigma]$.

Let $y(s^*) = y^*$ be any point of L within $\gamma_{s_0}(\varepsilon)$. Then

$$Z(s) = \frac{y^* - y_0}{s^* - s_0}(s - s_0) + y_0, \quad s_0 \leqq s < \infty$$

is the straight line from y_0 through y^*. We want to show that $Z(s)$ and $y(s)$ have at most n intersections within $\gamma_{s_0}(\varepsilon)$ i.e. there are at most n points s_i in I_0 for which $y(s_i) = Z(s_i)$. If there were more, then $F(s) = -Z(s) + y(s)$ would have at least $n + 1$ zeros i.e. $F(s_i) = 0$, $i = 1, 2, \ldots, n + 1$, s_i in I_0. Then the real-valued unction $F(s) \cdot A$ would have the same $n + 1$ zeros. By repeated use of Rolle's heorem $F^{(n)}(\bar{s}) \cdot A = 0$ for some $\bar{s}$ in $I_0(s_0)$. But since $n \geqq 2$

$$F^{(n)}(\bar{s}) \cdot A = y^{(n)}(s) \cdot A = 0.$$

But $|A| = 1$ and A is not perpendicular to $y^{(n)}(s)$ for s in I_0. Thus

$$y^{(n)}(\bar{s}) = 0.$$

But this is a contradiction since $|y^{(n)}(s) - y_0^{(n)}| < |y_0^{(n)}|$ for s in I_0, since then $y(s)$ is in $\gamma_{s_0}(\varepsilon)$. Thus in $\gamma_{s_0}(\varepsilon)$, L intersects each radius of $\gamma_{s_0}(\varepsilon)$ in at most n points. In particular, the extended radius r_a or r_b of $\gamma_{s_0}(\varepsilon)$ that passes through L_a or L_b meets L in at most n points inside $\gamma_{s_0}(\varepsilon)$.

If r_a or r_b meets L inside $\gamma_{s_0}(\varepsilon)$, put a concentric circle centered at y_0 that passes through that point of intersection of L with r_a or r_b which is closest to y_0. Let the circle be $\gamma_{s_0}(\bar{\varepsilon})$. If r_a and r_b do not meet L, let $\gamma_{s_0}(\bar{\varepsilon}) = \gamma_{s_0}(\varepsilon)$. Then if t is any point of L interior to $\gamma_{s_0}(\bar{\varepsilon})$, the line through y_0 and t does not contain L_a or L_b, and thus (C_2) is satisfied. Since y_0 was arbitrary, L belongs to $\mathscr{L}$ and the theorem is proved.

To find S^+ for L in $\mathscr{L}$ we take advantage of Theorem 2 and find the envelope E_a^+ of the circles passing through L_a, centered on L whose sides are on the left of L. We find then a similar envelope E_b^+ for the circles passing through L_b. Let C be the circle centered on L and passing through L_a and L_b. Let L^+ be the arc from L_a along E_a^+ or C to $E_a^+ \cap C$ then along C to $C \cap E_b^+$ then along E_b^+ or C to L_b, where the proper choices are the ones that give the maximum region between L and L^+. This maximum region is E^+, by definition, and as seen in Theorem 1, $E^+ = S^+$, since L belongs to $\mathscr{L}$. Note that L^+ may contain just part of C.

Let L be given in the α, β plane by $\beta = m(\alpha)$. There is no loss of generality in assuming L_a is at the origin and $m(0)=0$. Then the circles $C(\alpha)$ centered on L and passing through L_a are given by:

1. $$f(x,y,\alpha) = (x-\alpha)^2 + [y - m(\alpha)]^2 - \alpha^2 - m^2(\alpha) = 0$$

where (x,y) are generic points of $C(\alpha)$. To find the envelope, we eliminate α between 1. and

2. $$f_\alpha(x,y,\alpha) = x + m'(\alpha)y = 0.$$

The circles through $L_b = (x_0, y_0)$ are given by:

3. $$g(x,y,\alpha) = (x-\alpha)^2 + [y - m(\alpha)]^2 - (x_0 - \alpha)^2 - [y_0 - m(\alpha)]^2 = 0$$

and the envelope of these circles is given by eliminating α between 3. and

4. $$g_\alpha(x,y,\alpha) = (x - x_0) + m'(\alpha)(y - y_0) = 0$$

EXAMPLE. Let L be the arc of the parabola $\beta = \alpha^2$ between (0,0) and (1,1) with the direction from $L_a = (1,1)$ to $L_b = (0,0)$ In this case we get a quadratic equation for E_b^+ that can be solved to give:

$$x = \frac{1 - \sqrt{2(1-y)^6(2y+3)}}{2y-1} \quad \text{for } \tfrac{1}{2} < y \leqq 1,$$

and E_a^+ is given by:

$$x = \sqrt{\frac{-2y^3}{2y+1}} \quad \text{for } -\tfrac{1}{2} < y \leqq 0\,.$$

REFERENCE

1. Muskhelishvili, N. I., *Singular Integral Equations*, Translation by J. R. M. Radok, P. Noordhoff N. V. Groningen — Holland.

UNIVERSITY OF CALIFORNIA,
SANTA BARBARA, CALIFORNIA